AN EQUATION FOR THE EFFECTIVE MOMENT OF INERTIA FOR FRP-REINFORCED CONCRETE BEAMS

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Abstract
The deflection of concrete beams reinforced with GFRP bars is higher than those which are reinforced with steel bars. The purpose of this paper is to propose a new equation for estimating the effective moment of inertia in FRP-reinforced concrete beams. This will be accomplished by using experimental results and the genetic algorithm approach. Genetic algorithm is used for optimizing the error function between the experimental and analytical values obtained for the effective moment of inertia. Experimental data points were obtained from forty six FRP-reinforced concrete beams tested by different researchers and nine GFRP-reinforced concrete beams tested by the authors. These data points were selected as discrete points on the load-displacement curves of fifty five test specimens. These were between cracking and service levels of loading. Studies show that the intensity of the load, the reinforcement ratio and the elastic modulus of the FRP bars are most important parameters in calculating the deflection. In this paper, the effects of all aforementioned variables on the effective moment of inertia are taken into account. The results show that deflections predicted by the proposed equation are in good agreement with experimental values.

Keywords: Deflection, Effective moment of inertia, FRP bars, Reinforced concrete beams.

1. Introduction
Fiber-Reinforced Polymer (FRP) bars are used as a reinforcing material because of their corrosion resistant property. The behavior of concrete beams reinforced with FRP bars is different from that of steel reinforced concrete beams and is highly dependent on the type of fiber. FRP bars have high tensile strengths and appropriate durability. However, these bars have a linear elastic behavior up to the point of failure and therefore, are not categorized as a ductile material. On the other hand, FRP-reinforced concrete members have a relatively small stiffness value after cracking because the elastic modulus of FRP bars is typically lower than that of steel bars. For example, the elastic modulus of GFRP bars is between 20 and 25 percent of that in steel bars. Due to the low elastic modulus of GFRP bars, the deflection criterion may control the design of medium- and long-span beams reinforced with GFRP bars. Consequently, a method is needed to predict the expected service load deflections of FRP-reinforced members with reasonable accuracy.
The objective of this paper is to investigate the experimental and analytical deflections of concrete beams reinforced with FRP bars. In the paper, based on experimental results, a new equation is proposed for the effective moment of inertia of these beams. Three hundred and fifty data points are used to obtain this equation. In the development of this equation, a genetic algorithm optimization approach is used to minimize the difference between experimental deflections and calculated values.

2. Deflection Calculation

In a four-point load system, the maximum deflection ($\delta_{\text{max}}$) at the center of the beam can be calculated by:

$$\delta_{\text{max}} = \frac{P L^4}{48 E_c I_e} \left(3L^2 - 4L^2_a\right)$$

(1)

where, $L$ is the span of the beam, $P$ is the total concentrated load which is divided into two loads of $P/2$, each applied at a distance of $L_d$ from the support, $E_c$ is the elastic modulus of the concrete and $I_e$ is the effective moment of inertia of the beam section after cracking. Since the elastic modulus of GFRP bars is lower than that of steel bars, the stiffness of GFRP-reinforced concrete beams abruptly decreases when the applied moment exceeds $M_{cr}$. After cracking, the effective moment of inertia drops to a value slightly above $I_e$ [1]. According to ACI 318-05 [2] code provisions, the effective moment of inertia ($I_e$) suggested by Branson [3] can be determined as follows:

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right]I_{cr} \leq I_g$$

(2)

In Eq. (2), $M_{cr}$ is the cracking moment, $M_a$ is the maximum applied moment, $I_{cr}$ is the moment of inertia of the cracked transformed section and $I_g$ is the moment of inertia of the gross section. Studies show that Eq. (2) underestimates the deflections of FRP-reinforced concrete beams. Based on experimental results of studies on GFRP-reinforced concrete beams, Benmokrane et al. [4] modified Branson’s equation (Eq. 2). The modified equation is given by Eq. (3) as follows:

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + 0.84 \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right]I_{cr} \leq I_g$$

(3)

After considering the lower value of the elastic modulus and the bond properties of FRP bars, ACI 440.1R-03 [5] suggested a modified expression for the effective moment of inertia. This expression is given by Eq. (4) as follows:

$$I_e = \beta_d I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right]I_{cr} \leq I_g$$

(4)

The parameter $\beta_d$ accounts for the bond properties and elastic modulus of FRP bars and is given by:

$$\beta_d = \alpha_b \left(\frac{E_f}{E_s} + 1\right)$$

(5)

where $E_f$ is the elastic modulus of the FRP bars, $E_s$ is the elastic modulus of the reinforcing steel bars and $\alpha_b$ is a bond-dependent coefficient. According to ACI 440.1R-03 [5], the value of $\alpha_b$ can be taken as 0.5 for GFRP bars. Some studies have shown that the effective moment of inertia is affected by the relative reinforcement ratio ($\rho_f/\rho_h$) [6]. Based on the results of
GFRP reinforced concrete beam specimens tested by Yost et al. [6], $a_b$ must be significantly reduced to below the value of 0.5 recommended by ACI 440.1R-03 [5]. The results also show that this parameter depends on $\rho_f/\rho_{fb}$. The parameter $a_b$ is obtained by the linear regression analysis of the test results as follows:

$$a_b = 0.064 \left( \frac{\rho_f}{\rho_{fb}} \right) + 0.13$$  \hspace{1cm} (6)

In the above equation, $\rho_f$ is the reinforcement ratio and $\rho_{fb}$ is the balanced reinforcement ratio. Based on an evaluation of experimental results from several studies, a new expression for $\beta_d$, based on the relative reinforcement ratio, is given by ACI 440-06 [7] as follows:

$$\beta_d = 1 \left( \frac{\rho_f}{\rho_{fb}} \right) \leq 1$$  \hspace{1cm} (7)

A modification to the ACI 440.1R-06 [7] method for calculating the effective moment of inertia was proposed by Rafi and Nadjai [8] for all types of FRP bar as follows:

$$I_e = \left( \frac{M_{cr}}{M_a} \right)^3 \beta_d I_s + \frac{I_{cr}}{\gamma} \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right] \leq I_s$$  \hspace{1cm} (8)

In Eq. (8), the coefficient $\beta_d$ has a similar definition to the expression used by the ACI 440.1R-06 [7] code. Coefficient $\gamma$ is a relation obtained by the linear regression analysis of the test results and is given by Eq. (9) as follows:

$$\gamma = \left( 0.0017 \frac{\rho_f}{\rho_{fb}} + 0.8541 \right) \left( 1 + \frac{E_f}{2E_c} \right)$$  \hspace{1cm} (9)

Based on experimental results, Alsayed et al. [1] proposed two models for the effective moment of inertia. In the first model (Model A), the average value of the power ($m$) for GFRP reinforced concrete beams can be taken approximately 5.5 instead of 3 which is used in Branson’s equation. Another model (Model B) was suggested based on the regression analysis of the experimental results of $I_e/I_{cr}$ versus $M_a/M_{cr}$ for GFRP-reinforced beams. This model is given by the following equation:

$$I_e = \left[ 1.4 - \frac{2}{15} \left( \frac{M_a}{M_{cr}} \right) \right] I_{cr} \hspace{0.5cm} \text{for} \hspace{0.5cm} 1 < \frac{M_a}{M_{cr}} < 3 \hspace{0.5cm} \& \hspace{0.5cm} I_e = I_{cr} \hspace{0.5cm} \text{for} \hspace{0.5cm} \frac{M_a}{M_{cr}} > 3$$  \hspace{1cm} (10)

Fundamental concepts of tension stiffening were used by Bischoff [9] to modify Branson’s equation for the effective moment of inertia $I_e$. This modification is given by Eq. (11) as follows:

$$I_e = \frac{I_{cr}}{1 - \left( \frac{I_{cr}}{I_s} \right)^2 \left( \frac{M_{cr}}{M_a} \right)^2}$$  \hspace{1cm} (11)

The ISIS Canadian design manual [10] suggests that the effective moment of inertia for FRP reinforced concrete beams can be taken as:

$$I_e = \frac{I_s I_{cr}}{I_{cr} + \left[ 1 - 0.5 \left( \frac{M_{cr}}{M_a} \right)^2 \right] (I_s - I_{cr})}$$  \hspace{1cm} (12)
The Canadian Design Standard S806 [11] suggests that the moment-curvature method for calculating the deflection is well suited for FRP reinforced members because the moment-curvature diagram is estimated by two linear regions. CSA S806-02 [11] uses a simple equation for calculating the deflection of simply supported FRP-reinforced concrete beams under four-point bending:

\[
\delta_{\text{max}} = \frac{P L^2}{48 E I} \left( 3L^2 - 4L_a^2 - 8 \left( 1 - \frac{I_a}{I_g} \right) \left( \frac{M_{cr}}{M_a^3} \right)^3 I_a^2 \right) \quad (13)
\]

The CSA S806-02 [11] approach overestimates deflection considerably because this approach ignores the tension stiffening effect.

3. Proposed Model

In this paper, wide ranges of test data including three hundred and fifty experimental data points are used. These data points were obtained from load-displacement relationships of fifty five FRP-reinforced concrete beams. Nine of the specimens were tested by the authors and forty six FRP-reinforced concrete beams were tested by other researchers ([11], [6], [8], [12-14]). According to the experimental results and optimization by the genetic algorithm approach, Branson’s equation was modified to predict deflections closer to experimental values. Having the values of mid-span displacement and its corresponding load, the experimental values of the effective moment of inertia is calculated by Eq. (14) as follows:

\[
(I_e)_{\text{exp}} = \frac{P_{\text{exp}} L_a}{48 E \delta_{\text{exp}}} \left( 3L^2 - 4L_a^2 \right) \quad (14)
\]

where \( P_{\text{exp}} \) is the experimental load and \( \delta_{\text{exp}} \) is the experimental mid-span displacement corresponding to \( P_{\text{exp}} \). Similar to Branson’s equation (Eq. 2), the effective moment of inertia can be given by Eq. (15) in a general form as follows:

\[
(I_e)_{\text{exp}} = \left( \frac{M_{cr}}{M_a} \right)^m I_g + \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^m \right] I_{cr} \quad \Rightarrow \quad \frac{(I_e)_{\text{exp}} - I_{cr}}{I_g - I_{cr}} = \left( \frac{M_{cr}}{M_a} \right)^m \quad (15)
\]

The value of \( m \) can be obtained using \((I_e)_{\text{exp}}\) and \( M_{cr}/M_a \) for every experimental data point at any levels of loading:

\[
m = \log \left( \frac{(I_e)_{\text{exp}} - I_{cr}}{I_g - I_{cr}} \right) / \log \left( \frac{M_{cr}}{M_a} \right) \quad (16)
\]

The experimental values of \( m \) show that power \( m \) is not constant. This parameter decreases with increase in levels of loading (decrease in \( M_{cr}/M_a \) values) and \( \rho_f/\rho_b \) values. The effects of the elastic modulus of FRP bars, reinforcement ratio and level of loading on the power of \( m \) in Branson’s equation are taken into account in this study. The influences of different parameters are introduced by the coefficients \( X_I \) to \( X_6 \) in the following equations:

\[
(I_e)_{\text{exp}} = X_5 \left( \frac{M_{cr}}{M_a} \right)^m I_g + X_6 \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^m \right] I_{cr} \quad (17)
\]

The power of \( m \) is given by Eq. (18).

\[
m = X_1 + X_2 \frac{\rho}{\rho_b} + X_3 \frac{M_{cr}}{M_a} + X_4 \frac{E_f}{E_s} \quad (18)
\]
The optimization variables \((X_1 \text{ to } X_6)\), which determine the influence of different parameters in the beam deflection, are calculated by minimizing the objective function in genetic algorithm method. The objective function is defined by Eq. (19).

\[
e = \left( \frac{(I_e)_{\text{theo}} - (I_e)_{\exp}}{(I_e)_{\exp}} \right) \times 100
\] (19)

The genetic algorithm method (GA) is an optimization and search technique based on the principles of genetics and natural selection. The values of \(X_1 \text{ to } X_6\) obtained by the optimization are as follows:

\[
(I_e)_{\text{prop}} = 0.13 \left( \frac{M_{cr}}{M_a} \right)^{0.3} I_g + 0.89 \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^{0.3} \right] I_{cr} \leq I_g
\] (20)

\[
m = -0.24 \frac{\rho_f}{\rho_b} + 5.35 \frac{M_{cr}}{M_a} + 2.28 \frac{E_f}{E_s}
\] (21)

4. Experimental Program

Nine beam specimens with the dimensions of \(150 \times 200 \times 2300\) (mm) and shear span of 700 (mm) were manufactured and tested. All specimens with an effective span length of 2000 mm were subjected to a four-point flexural test. Two concentrated loads were applied to the specimens by means of a hydraulic jack and a spreader beam (Fig. 1). A load cell was set directly under the hydraulic jack and on the top of the spreader beam to transfer the load increments to a Data Logger acquisition system. A LVDT (Linear Variable Displacement Transducer) was set at the centre of the specimens to transfer the mid-span displacement values to the Data Logger. The load increments and the corresponding displacements could be transferred from the Data Logger to the computer system.

![Figure 1. Details of test apparatus](image)

Three different concretes with compressive strengths of 20, 38 and 64MPa were used for the beam specimens. Two 10-mm-diameter steel bars were used as compression reinforcement. Steel stirrups with a diameter of 8 mm and spacing of 80 mm were used for all specimens to provide shear strength. The structural details of the specimens are shown in Table 1. The GFRP bars used for the specimens were sand-coated with an ultimate tensile strength and modulus of elasticity of 700 MPa and 41 GPa, respectively. Test specimens had three different concrete compressive strengths and reinforcement ratios that produced a wide range of relative reinforcement ratios \((\rho_f / \rho_b)\) between 0.51 and 7.75. Using the results of these specimens, the effect of \(\rho_f / \rho_b\) on the proposed equations for deflection prediction can be taken into account.

| Table 1. Details of test specimens |
Longitudinal Reinforcement (GFRP)

<table>
<thead>
<tr>
<th>Specimen number</th>
<th>$f'_c$ (MPa)</th>
<th>$(\rho_f/\rho_{fb})$</th>
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<th>$(\rho_f/\rho_{fb})$</th>
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<th>$f'_c$ (MPa)</th>
<th>$(\rho_f/\rho_{fb})$</th>
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</thead>
<tbody>
<tr>
<td>B1</td>
<td>20</td>
<td>1.64</td>
<td>B2</td>
<td>20</td>
<td>5.44</td>
<td>B3</td>
<td>20</td>
<td>7.75</td>
</tr>
<tr>
<td>B4</td>
<td>38</td>
<td>0.87</td>
<td>B5</td>
<td>38</td>
<td>2.88</td>
<td>B6</td>
<td>38</td>
<td>4.11</td>
</tr>
<tr>
<td>B7</td>
<td>64</td>
<td>0.51</td>
<td>B8</td>
<td>64</td>
<td>1.70</td>
<td>B9</td>
<td>64</td>
<td>2.42</td>
</tr>
</tbody>
</table>

Ultimate experimental load and ultimate mid-span displacement of test specimens are shown in Table 2.

<table>
<thead>
<tr>
<th>Specimen number</th>
<th>$f'_c$ (MPa)</th>
<th>$(\rho_f/\rho_{fb})$</th>
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<th>$(\rho_f/\rho_{fb})$</th>
</tr>
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<tbody>
<tr>
<td>B1</td>
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<td>1.64</td>
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<td>B3</td>
<td>20</td>
<td>7.75</td>
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<tr>
<td>B4</td>
<td>38</td>
<td>0.87</td>
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<td>38</td>
<td>2.88</td>
<td>B6</td>
<td>38</td>
<td>4.11</td>
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<tr>
<td>B7</td>
<td>64</td>
<td>0.51</td>
<td>B8</td>
<td>64</td>
<td>1.70</td>
<td>B9</td>
<td>64</td>
<td>2.42</td>
</tr>
</tbody>
</table>

Table 2. Ultimate load and displacement of test specimens

<table>
<thead>
<tr>
<th>Specimen number</th>
<th>Ultimate load (kN)</th>
<th>Ultimate displacement (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>33.5</td>
<td>71.2</td>
</tr>
<tr>
<td>B2</td>
<td>57.0</td>
<td>43.8</td>
</tr>
<tr>
<td>B3</td>
<td>56.4</td>
<td>36.8</td>
</tr>
<tr>
<td>B4</td>
<td>32.9</td>
<td>64.8</td>
</tr>
<tr>
<td>B5</td>
<td>86.0</td>
<td>64.8</td>
</tr>
<tr>
<td>B6</td>
<td>95.4</td>
<td>61.2</td>
</tr>
<tr>
<td>B7</td>
<td>38.0</td>
<td>47.8</td>
</tr>
<tr>
<td>B8</td>
<td>96.4</td>
<td>63.9</td>
</tr>
<tr>
<td>B9</td>
<td>112.1</td>
<td>69.5</td>
</tr>
</tbody>
</table>

5. Comparison of Results

Figures 2 to 7 compare experimental deflections with the values predicted by analytical models at serviceability limit state. These figures show that the deflections predicted by proposed model correlate well with the experimental values in comparison with other models.
A statistical study on the ratio of calculated to experimental deflection values ($\frac{\delta_{cal}}{\delta_{exp}}$) is carried out to evaluate the accuracy of the proposed equation and the previous methods of deflection calculation. The average and the standard deviation for all of the data, the high levels of loading ($M_d/M_t \geq 0.25$) and the high relative reinforcement ratios ($\rho_f/\rho_{fb} \geq 3$) are shown in Table 3. According to this table, the prediction of the proposed equation for deflection in terms of the mean value and the standard deviation is satisfactory. Results show that the Canadian design manual ISIS [10] and the CSA-S806-02 [11] design code, estimate the deflection more conservatively than the ACI 440.1R-06 [7] code.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Total</th>
<th>High Levels of Loading $M_d/M_t \geq 4$</th>
<th>High reinforcement ratios $\rho_f/\rho_{fb} \geq 3$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Standard Deviation</td>
<td>Mean</td>
</tr>
<tr>
<td>ACI 318-05 [2]</td>
<td>0.569</td>
<td>0.276</td>
<td>0.870</td>
</tr>
<tr>
<td>ACI 440.1R-03 [5]</td>
<td>0.673</td>
<td>0.266</td>
<td>0.894</td>
</tr>
<tr>
<td>ACI 440.1R-06 [7]</td>
<td>0.794</td>
<td>0.244</td>
<td>0.911</td>
</tr>
<tr>
<td>Benmokrane et al. [4]</td>
<td>1.166</td>
<td>0.344</td>
<td>1.106</td>
</tr>
<tr>
<td>Yost et al. [6]</td>
<td>0.999</td>
<td>0.290</td>
<td>0.929</td>
</tr>
<tr>
<td>ISIS Canada [10]</td>
<td>1.030</td>
<td>0.310</td>
<td>0.917</td>
</tr>
<tr>
<td>Alsayed et al. [1]– Model A</td>
<td>0.821</td>
<td>0.313</td>
<td>0.933</td>
</tr>
<tr>
<td>Alsayed et al. [1]– Model B</td>
<td>0.951</td>
<td>0.336</td>
<td>0.935</td>
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<td>Bischoff [9]</td>
<td>0.861</td>
<td>0.252</td>
<td>0.904</td>
</tr>
<tr>
<td>Rafi and Nadjai [8]</td>
<td>0.783</td>
<td>0.236</td>
<td>0.913</td>
</tr>
<tr>
<td>CSA S806-02 [11]</td>
<td>1.067</td>
<td>0.314</td>
<td>0.931</td>
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<tr>
<td>Proposed equation</td>
<td>1.122</td>
<td>0.305</td>
<td>0.965</td>
</tr>
</tbody>
</table>

As shown in Table 3, the mean value and the standard deviation of $\frac{\delta_{cal}}{\delta_{exp}}$ ratio for the proposed model are respectively 0.965 and 0.157, for high levels of loading. Also, these values are 1.142 and 0.237 for high relative reinforcement ratios. Therefore, in higher levels of loading and relative reinforcement ratios, the proposed model is more accurate than other models. As shown in Table 3, the equation given by ACI 440.1R-06 [7] has an average of 0.794 and hence considerably underestimates the deflection of beams. The mean and the standard deviation values predicted by Yost et al. [6] and ISIS Canada [10] models are (0.999, 0.290) and (1.030, 0.310), respectively. Therefore, these models satisfactorily predict the deflection. Figures 2 to 7 and Table 3 show that the deflections predicted by Bischoff [9] and ACI 440.1R-06 [7] equations are lower than the experimental results while the values calculated by Benmokrane et al. [4] equation are higher than the experimental deflections. In addition, the deflections calculated by the proposed model, CSA S806-02 [11] and the Canadian design manual ISIS [10] have good agreement with the experimental values.

6. Conclusions

Based on the present study, the following conclusions can be drawn:

1- The effects of the reinforcement ratio and elastic modulus of FRP bars are considered in the Yost equation. The deflections estimated by Yost’s model are more accurate than those predicted by ACI 440.1R-03 and ACI 440.1R-06 provisions.

2- ACI 440.1R-03 and ACI 440.1R-06 codes underestimate the deflections while the values calculated by the CSA code and Canadian design manual ISIS are conservative.

3- The proposed equation accounts for the most effective parameters such as the elastic modulus of FRP bars, relative reinforcement ratio, levels of loading for calculating the
deflection. Therefore, a variety of effective parameters are taken into account in the proposed equations. The influence of the aforementioned parameters is determined by using the genetic algorithm optimization method. The values predicted by the proposed equation correlate well with the experimental results especially for the cases of high levels of loading and high reinforcement ratios.

4- The proposed model, Yost’s equation and the Canadian Design Manual (ISIS) expression have the best average and standard deviation of $\delta_{cal}/\delta_{exp}$ compared to other methods.

7. References


